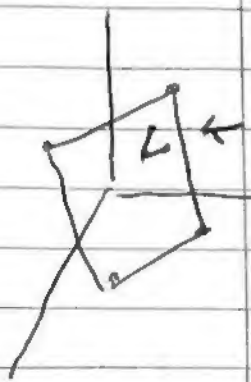# Calc 3   1, 9, 21

Dot Product

a. $\vec{v} \cdot \vec{u} = 0$ when orthogonal

12.4: Cross product
* Everything today is in $\mathbb{R}^3$
Goal - Given two vectors, construct a third
(ideally orthogonal to both)

To find a vector perpendicular to
the plane, we find one perpendicular
to the two vectors on the plane.

How?! Let $\vec{u} = (u_1, u_2, u_3)$, $\vec{v} = (v_1, v_2, v_3)$

and desired vector $\vec{w} = (w_1, w_2, w_3)$.
$\vec{w}$ must be orthogonal to both $\vec{u}$ and
$\vec{v}$.

① $0 = \vec{w} \cdot \vec{u} = w_1 u_1 + w_2 u_2 + w_3 u_3$
② $0 = \vec{w} \cdot \vec{v} = w_1 v_1 + w_2 v_2 + w_3 v_3$

Multiply eq. 1 by v_3 and eq. 2 by u_3
to get two new equations

①* $0 = w_1(u_1 v_3) + w_2(u_2 v_3) + w_3(u_3 v_3)$
②* $0 = w_1(v_1 u_3) + w_2(u_3 v_2) + w_3(u_3 v_3)$

Subtracting 2^* from 1^*

$0 = v_3(\vec{w} \cdot \vec{u}) - u_3(\vec{w} \cdot \vec{v})$
$0 = w_1(u_1 v_3 - u_3 v_1) + w_2(u_2 v_3 - u_3 v_2)$
$\quad = -w_1(-u_1 v_3 + u_3 v_1) + w_2(u_2 v_3 - u_3 v_2)$

Solution: $w_1 = u_2 v_3 - u_3 v_2$
$\qquad w_2 = -(u_1 v_3 - u_3 v_1)$

Plugging in to initial problem
yields $w_3 = u_1 v_2 - u_2 v_1$

Aside easiest solution
to $-ax + by = 0$ is
$x = b, y = a$

The solution assumes $u_3 \neq 0$

$$\vec{w} = (w_1, w_2, w_3$$
$$= (u_2 v_3 - u_3 v_2, \ -(u_1 v_3 - u_3 v_1), \ u_1 v_2 - u_2 v_1)$$

Exercise ⟶ To verify, check if dot products $= 0$

⟶ Linear Algebra approach - determinants
Def. The determinant of a 2×2
matrix is $ad - bc$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

& Det of a 3×3 matrix is

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

$$a\left(\det\begin{bmatrix} e & f \\ h & i \end{bmatrix}\right) - b\left(\det\begin{bmatrix} d & f \\ g & i \end{bmatrix}\right) + c\left(\det\begin{bmatrix} d & e \\ g & h \end{bmatrix}\right)$$

ex. $\det\begin{bmatrix} 1 & -2 & 3 \\ -1 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ $\quad 1\left(\det\begin{pmatrix} 1 & -2 \\ -1 & -1 \end{pmatrix}\right) = (-1 + 2) = 1$

The vector is a symbolic <u>determinant</u>

$$\vec{u} \rightarrow \begin{vmatrix} i & j & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \rightarrow i(u_2 v_3 - v_2 u_3) - j(u_1 v_3 - v_1 u_3)$$
$$\vec{v} \rightarrow \qquad\qquad\qquad + k(u_1 v_2 - u_3 v_1)$$

$$\langle (u_2 v_3 - u_3 v_2), -(u_1 v_3 - v_1 u_3), (u_1 v_2 - u_3 v_1) \rangle$$

Cross product of $\vec{u}$ with $\vec{v}$ is ⤴
and there is orthogonal to both vectors.

Cross product uses two vectors in $\mathbb{R}^3$
to produce a vector in $\mathbb{R}^3$

Properties of the cross product
Let $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^3$, and $c \in \mathbb{R}$
(1) $\vec{u} \times \vec{v} = -(\vec{v} \times \vec{u})$
(2) $(c\vec{u}) \times \vec{v} = c(\vec{u} \times \vec{v})$
(3) $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$
(4) $(\vec{u} + \vec{b}) \times \vec{w} = \vec{u} \times \vec{w} + \vec{w}\vec{b}$
(5) $\vec{u} \cdot (\vec{v} \times \vec{w}) = (\vec{u} \times \vec{v}) \cdot \vec{w}$
(6) $\vec{u} \times (\vec{v} \times \vec{w}) = (\vec{u} \cdot \vec{w})\vec{v} - (\vec{u} \cdot \vec{v})\vec{w}$

Geometric properties

(1) $\vec{u} \times \vec{v}$ is orthogonal to both vectors
(2) $|\vec{u} \times \vec{v}|$ is $|\vec{u}| \cdot |\vec{v}| \sin(\theta)$
(3) parallel if cross product $= \vec{0}$